

EXISTENCE OF NON-TRIVIAL SOLUTIONS FOR NONLINEAR FRACTIONAL SCHRÖDINGER-POISSON EQUATIONS

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ABSTRACT. We study the nonlinear fractional Schrödinger-Poisson equations

$$\begin{cases} (-\Delta)^s u + u + \phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0, 1]$, $2t + 4s > 3$. Under some assumptions on f , we obtain the existence of non-trivial solutions. The proof is based on the perturbation method and the mountain pass theorem.

1. INTRODUCTION

In this paper, we are concerned with the existence of non-trivial solutions for the following fractional Schrödinger-Poisson equation

$$(1.1) \quad \begin{cases} (-\Delta)^s u + u + \phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0, 1]$, $2t + 4s > 3$, $(-\Delta)^s$ denotes the fractional Laplacian.

When $s = t = 1$, the equation (1.1) reduces to Schrödinger-Poisson equation, which describes system of identical charged particles interacting each other in the case where magnetic effects can be neglected [3, 6]. When $\phi = 0$, (1.1) reduces to a fractional Schrödinger equation, which is a fundamental equation in fractional quantum mechanics [7, 8].

Recently, some authors proposed a new approach called perturbation method to study the quasilinear elliptic equations, see [10]. The idea is to get the existence of critical points of the perturbed energy functional I_λ for $\lambda > 0$ small and then taking $\lambda \rightarrow 0$ to obtain solutions of original problems. Very recently, Feng [1] used the perturbation method to study the Schrödinger-Poisson equation

$$(1.2) \quad \begin{cases} -\Delta u + u + \phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\alpha/2} \phi = u^2, \quad \lim_{|x| \rightarrow \infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\alpha \in (1, 2]$. Under some conditions, the problem (1.2) possesses at least a nontrivial solution.

We point out that when $s = 1$ and $t \in (\frac{1}{2}, 1]$, the problem (1.1) boils down to (1.2). The main result of this paper is described as follows.

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Theorem 1.1. *Suppose f satisfies the following conditions:*

(A1) *For every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exist constants $C_1 > 0$ and $p \in [2, 2_s^*)$ such that*

$$|f(x, u)| \leq C_1(|u| + |u|^{p-1}),$$

where $2_s^ = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent;*

(A2) *$f(x, u) = o(|u|)$, $|u| \rightarrow 0$, uniformly on \mathbb{R}^3 ;*

(A3) *there exists $\mu > 4$ such that*

$$0 < \mu F(x, u) \leq u f(x, u)$$

holds for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R} \setminus \{0\}$, where $F(x, u) = \int_0^u f(x, s) ds$;

Then problem (1.1) has at least a nontrivial solution.

The paper is organized as follows. In Section 2, we present some preliminaries results. In Section 3, we will prove Theorem 1.1.

2. PRELIMINARIES

For $p \in [1, \infty)$, we denote by $L^p(\mathbb{R}^3)$ the usual Lebesgue space with the norm $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$. For any $p \in [1, \infty)$ and $s \in (0, 1)$, we recall some definitions of fractional Sobolev spaces and the fractional Laplacian $(-\Delta)^s$, for more details, we refer to [5]. $H^s(\mathbb{R}^3)$ is defined as follows

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}$$

with the norm

$$(2.1) \quad \|u\|_{H^s} = (|\mathcal{F}u(\xi)|^2 + |\xi|^{2s} |\mathcal{F}u(\xi)|^2)^{\frac{1}{2}},$$

where $\mathcal{F}u$ denotes the Fourier transform of u . By $\mathcal{S}(\mathbb{R}^n)$, we denote the Schwartz space of rapidly decaying C^∞ functions in \mathbb{R}^n . For $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, $(-\Delta)^s$ is defined by

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s} (\mathcal{F}f)), \quad \forall \xi \in \mathbb{R}^n.$$

By Plancherel's theorem, we have $\|\mathcal{F}u\|_2 = \|u\|_2$, $\|\xi|^s \mathcal{F}u\|_2 = \|(-\Delta)^{\frac{s}{2}} u\|$. Then by (2.1), we get the equivalent norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2) dx \right)^{\frac{1}{2}}.$$

For $s \in (0, 1)$, the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as follows

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : |\xi|^s \mathcal{F}u(\xi) \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{s,2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Lemma 2.1. (Theorem 2.1 in [12]). *For any $s \in (0, \frac{3}{2})$, $D^{s,2}(\mathbb{R}^3)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^3)$, i.e., there exists $c_s > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx \right)^{2/2_s^*} \leq c_s \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad u \in D^{s,2}(\mathbb{R}^3).$$

We consider the variational setting of (1.1). From Theorem 6.5 and Corollary 7.2 in [5], it is known that the space $H^s(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$ for any $q \in [2, 2_s^*]$ and the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is locally compact for $q \in [1, 2_s^*)$.

If $2t + 4s > 3$, then $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. For $u \in H^s(\mathbb{R}^3)$, the linear operator $T_u : D^{t,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined as

$$T_u(v) = \int_{\mathbb{R}^3} u^2 v dx,$$

By Hölder inequality and Lemma 2.1,

$$(2.2) \quad |T_u(v)| \leq \|u\|_{12/(3+2t)}^2 \|v\|_{2_t^*} \leq C \|u\|_{H^s}^2 \|v\|_{D^{t,2}}.$$

Set

$$\eta(u, v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} u \cdot (-\Delta)^{\frac{t}{2}} v dx, \quad u, v \in D^{t,2}(\mathbb{R}^3).$$

It is clear that $\eta(u, v)$ is bilinear, bounded and coercive. The Lax-Milgram theorem implies that for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $T_u(v) = \eta(\phi_u^t, v)$ for any $v \in D^{t,2}(\mathbb{R}^3)$, that is

$$(2.3) \quad \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx.$$

Therefore, $(-\Delta)^t \phi_u^t = u^2$ in a weak sense. Moreover,

$$(2.4) \quad \|\phi_u^t\|_{D^{t,2}} = \|T_u\| \leq C \|u\|_{H^s}^2.$$

Since $t \in (0, 1]$ and $2t + 4s > 3$, then $\frac{12}{3+2t} \in (2, 2_s^*)$. From Lemma 2.1, (2.2) and (2.3), it follows that

$$(2.5) \quad \|\phi_u^t\|_{D^{t,2}}^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} \phi_u^t|^2 dx = \int_{\mathbb{R}^3} u^2 \phi_u^t dx \leq \|u\|_{\frac{12}{3+2t}}^2 \|\phi_u^t\|_{2_t^*} \leq C \|u\|_{\frac{12}{3+2t}}^2 \|\phi_u^t\|_{D^{t,2}}.$$

Then

$$(2.6) \quad \|\phi_u^t\|_{D^{t,2}} \leq C \|u\|_{\frac{12}{3+2t}}^2.$$

For $x \in \mathbb{R}^3$, we have

$$(2.7) \quad \phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy,$$

which is the Riesz potential [9], where

$$c_t = \frac{\Gamma(\frac{3-2t}{2})}{\pi^{3/2} 2^{2t} \Gamma(t)}.$$

Substituting ϕ_u^t in (1.1), we have the fractional Schrödinger equation

$$(2.8) \quad (-\Delta)^s u + u + \phi_u^t u = f(x, u), \quad x \in \mathbb{R}^3,$$

The energy functional $I : H^s(\mathbb{R}^3) \rightarrow \mathbb{R}$ corresponding to problem (2.8) is defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx.$$

It is easy to see that I is well defined in $H^s(\mathbb{R}^3)$ and $I \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$, and

$$(2.9) \quad \langle I'(u), v \rangle = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv + \phi_u^t uv - f(x, u)v) dx, \quad v \in H^s(\mathbb{R}^3).$$

Definition 2.2. (1) We call $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ is a weak solution of (1.1) if u is a weak solution of (2.8).

(2) We call u is a weak solution of (2.8) if

$$\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv + \phi_u^t uv - f(x, u)v) dx = 0,$$

for any $v \in H^s(\mathbb{R}^3)$.

Assume that the potential $V(x)$ satisfies the condition

(V) $V \in C(\mathbb{R}^3)$, $\inf_{x \in \mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a constant. For every $M > 0$, $meas\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$, where $meas(\cdot)$ denotes the Lebesgue measure in \mathbb{R}^3 .

Let

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(x)u^2) dx < \infty \right\}.$$

Then E is a Hilbert space with the inner product

$$\langle u, v \rangle_E = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv) dx$$

and the norm $\|u\|_E = \langle u, u \rangle_E^{\frac{1}{2}}$. By Lemma 2.3 in [4], it is known that E is compactly embedded in $L^p(\mathbb{R}^3)$ for $2 \leq p < 2_s^*$.

For fixed $\lambda \in (0, 1]$, we introduce the following inner product

$$\langle u, v \rangle_{H_\lambda^s} = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \lambda V(x)uv) dx$$

and the norm $\|u\|_{H_\lambda^s} = \langle u, u \rangle_{H_\lambda^s}^{\frac{1}{2}}$. Denote $E_\lambda = (E, \|\cdot\|_{H_\lambda^s})$.

Define the perturbed functional $I_\lambda : E \rightarrow \mathbb{R}$:

$$(2.10) \quad I_\lambda(u) = I(u) + \frac{\lambda}{2} \int_{\mathbb{R}^3} V(x)u^2 dx, \quad \lambda \in (0, 1].$$

Lemma 2.3. Suppose that $V(x) \geq 0$ and (A1), (A2) hold. Then there exist $\rho > 0$, $\eta > 0$ such that for fixed $\lambda \in (0, 1]$,

$$\inf_{u \in E, \|u\|_E = \rho} I_\lambda(u) > \eta,$$

where ρ and η are independent of λ .

Proof. By (A1) and (A2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R}.$$

Then

$$|F(x, u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p.$$

For $\rho > 0$, set

$$\Sigma_\rho = \{u \in E : \|u\|_E \leq \rho\}.$$

It is known that E is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2_s^*]$ ($2_s^* = \frac{6}{3-2s}$), then $\|u\|_q \leq C_0 \|u\|_E$. Since $p \in (2, 2_s^*)$, for $u \in \partial\Sigma_\rho$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|_E^2 + \frac{\lambda}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t(x) u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx \\ &\geq \frac{(1-\varepsilon)\rho^2}{2} - \frac{C_\varepsilon C_0}{p} \rho^p. \end{aligned}$$

For $\varepsilon \in (0, 1)$ and sufficiently small ρ , the conclusion holds. \square

Lemma 2.4. *Assume that (A3) and (A4) hold. Then there exists $e \in E$ with $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$ for fixed $\lambda \in (0, 1]$, where ρ is the same as in Lemma 2.3.*

Proof. By (A3), there exists a constant $C > 0$ such that

$$(2.11) \quad F(x, u) \geq C|u|^\mu, \quad u \in \mathbb{R}.$$

By (2.4), (2.5),

$$(2.12) \quad \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \|\phi_u^t\|_{D^{t,2}}^2 \leq C \|u\|_{H^s}^4.$$

For $\xi > 0$ and $v \in C_0^\infty(\mathbb{R}^3)$, by (2.10), (2.11) and (2.12), we have

$$\begin{aligned} I_\lambda(\xi v) &= \frac{\xi^2}{2} \|v\|_{H_\lambda^s}^2 + \frac{\xi^2}{2} \|v\|_2^2 + \frac{\xi^4}{4} \int_{\mathbb{R}^3} \phi_v^t v^2 dx - \int_{\mathbb{R}^3} F(x, \xi v) dx \\ &\leq \frac{\xi^2}{2} \|v\|_E^2 + \frac{\xi^2}{2} \|v\|_2^2 + \frac{C\xi^4}{4} \|v\|_{H^s}^4 - C\xi^\mu \|v\|_\mu^\mu \rightarrow -\infty \end{aligned}$$

as $\xi \rightarrow +\infty$. Define a path $h : [0, 1] \rightarrow E$ by $h(\eta) = \eta\xi v$. For ξ large enough, we get

$$\|h(1)\|_E = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} h(1)|^2 + V(x) h^2(1)) dx \right)^{\frac{1}{2}} > \rho \text{ and } I_\lambda(h(1)) < 0.$$

Choose $e = h(1)$, we obtain the conclusion. \square

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Assume that (V), (A1), (A3) hold. Then I_λ satisfies the Palais-Smale condition on E for fixed $\lambda \in (0, 1]$.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence in E , i.e., $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \rightarrow 0$. We will show that $\{u_n\}$ has a convergent subsequence in E . Then

$$\begin{aligned} (3.1) \quad C + \|u_n\|_E &\geq I_\lambda(u_n) - \frac{1}{\mu} \langle I'_\lambda(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_{H_\lambda^s}^2 + \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_2^2 + \left(\frac{1}{4} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{u_n f(x, u_n)}{\mu} - F(x, u_n) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \lambda \|u_n\|_E^2. \end{aligned}$$

This implies that $\{u_n\}$ is bounded in E .

Up to a subsequence, we assume that $u_n \rightharpoonup u$ in E . Since E is compactly embedded

in $L^p(\mathbb{R}^3)$ for $2 \leq p < 2_s^*$, then $u_n \rightarrow u$ in $L^p(\mathbb{R}^3)$, $2 \leq p < 2_s^*$. By (2.9), (2.10), we get

$$\begin{aligned} \|u_n - u\|_\lambda^2 &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle - \|u_n - u\|_2^2 - \int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u) dx \\ (3.2) \quad &+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx. \end{aligned}$$

Clearly, we have

$$(3.3) \quad \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle \rightarrow 0 \quad \text{and} \quad \|u_n - u\|_2^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the generalization of Hölder inequality, Lemma 2.1 and (2.6), it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \right| &\leq \|\phi_{u_n}^t\|_{2^*} \|u_n\|_{\frac{12}{3+2t}} \|u_n - u\|_{\frac{12}{3+2t}} \\ &\leq C \|\phi_{u_n}^t\|_{D^{t,2}} \|u_n\|_{\frac{12}{3+2t}} \|u_n - u\|_{\frac{12}{3+2t}} \\ &\leq C \|u_n\|_{\frac{12}{3+2t}}^3 \|u_n - u\|_{\frac{12}{3+2t}} \\ &\leq C \|u_n\|_E^3 \|u_n - u\|_{\frac{12}{3+2t}}. \end{aligned}$$

Similarly,

$$\left| \int_{\mathbb{R}^3} \phi_u^t u (u_n - u) dx \right| \leq C \|u\|_E^3 \|u_n - u\|_{\frac{12}{3+2t}}.$$

We have

$$(3.4) \quad \left| \int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u) dx \right| \leq \left| \int_{\mathbb{R}^3} \phi_{u_n}^t u_n (u_n - u) dx \right| + \left| \int_{\mathbb{R}^3} \phi_u^t u (u_n - u) dx \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By (A1), Hölder inequality and Minkowski inequality,

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \right| \\ &\leq C_1 \int_{\mathbb{R}^3} (|u_n| + |u|)|u_n - u| dx + C_1 \int_{\mathbb{R}^3} (|u_n|^{p-1} + |u|^{p-1})|u_n - u| dx \\ &\leq C_1 (\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + C_1 (\|u_n\|_p^{p-1} + \|u\|_p^{p-1}) \|u_n - u\|_p \\ &\leq C_1 (\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + C_1 (\|u_n\|_p^{p-1} + \|u\|_p^{p-1}) \|u_n - u\|_p \\ (3.5) \quad &\leq C (\|u_n\|_E + \|u\|_E) \|u_n - u\|_2 + C (\|u_n\|_E^{p-1} + \|u\|_E^{p-1}) \|u_n - u\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By (3.2), (3.3), (3.4) and (3.5), we see that $\{u_n\}$ converges strongly in E for fixed $\lambda \in (0, 1]$, therefore I_λ satisfies the Palais-Smale condition on E for fixed $\lambda \in (0, 1]$. \square

Theorem 3.2. *Assume that (A3) hold. Let $\lambda_n \rightarrow 0$ and let $\{u_n\} \subset E$ be a sequence of critical points of I_{λ_n} satisfying $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq C$ for some C independent of n . Then up to a subsequence as $n \rightarrow \infty$, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, u is a critical point of I .*

Proof. By $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq C$, we have

$$\begin{aligned}
C &\geq I_{\lambda_n}(u_n) - \frac{1}{\mu} \langle I'_{\lambda_n}(u_n), u_n \rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_2^2 + \left(\frac{1}{4} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \\
&\quad + \int_{\mathbb{R}^3} \left(\frac{u_n f(x, u_n)}{\mu} - F(x, u_n) \right) dx \\
(3.6) \quad &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 dx.
\end{aligned}$$

Then up to a subsequence, we have $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. By Lemma 2.3 in [2], $\phi_{u_n}^t \rightharpoonup \phi_u^t$ in $D^{t,2}(\mathbb{R}^3)$. Taking $v \in C_0^\infty(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u v dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t u v dx, \text{ as } n \rightarrow \infty.$$

From the generalization of Hölder inequality, it follows that

$$\left| \int_{\mathbb{R}^3} \phi_{u_n}^t (u_n - u) v dx \right| \leq \|\phi_{u_n}^t\|_{2_t^*} \|u_n - u\|_{L^{12/(3+2t)}(\Omega)} \|v\|_{L^{12/(3+2t)}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where Ω is the support of v . Then,

$$\left| \int_{\mathbb{R}^3} \phi_{u_n}^t u_n v dx - \int_{\mathbb{R}^3} \phi_u^t u v dx \right| \leq \left| \int_{\mathbb{R}^3} (\phi_{u_n}^t - \phi_u^t) u v dx \right| + \left| \int_{\mathbb{R}^3} \phi_{u_n}^t (u_n - u) v dx \right| \rightarrow 0$$

as $n \rightarrow \infty$, for all $v \in C_0^\infty(\mathbb{R}^3)$.

By (2.9), (2.10),

$$\langle I'_{\lambda_n}(u_n), v \rangle = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} v + u_n v + \phi_{u_n}^t u_n v - f(x, u_n) v) dx + \lambda_n \int_{\mathbb{R}^3} V(x) u_n v dx,$$

where $v \in C_0^\infty(\mathbb{R}^3)$. By (3.6), Hölder inequality,

$$\lambda_n \int_{\mathbb{R}^3} V(x) u_n v = \lambda_n \int_{\mathbb{R}^3} (\sqrt{V(x)} u_n) (\sqrt{V(x)} v) dx \leq \lambda_n^{1/2} \left(\int_{\mathbb{R}^3} \lambda_n V(x) v^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^3} V(x) u_n^2 dx \right)^{1/2}$$

$\rightarrow 0$ as $n \rightarrow \infty$. Thus, we see that $I'(u)v = 0$ for all $v \in C_0^\infty(\mathbb{R}^3)$. It is known that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^s(\mathbb{R}^3)$, see Theorem 2.4 in [5]. Therefore, $I'(u)v = 0$ for all $v \in H^s(\mathbb{R}^3)$, u is a critical point of I . \square

Lemma 3.3. (Lemma 2.3 in [11]). *Let $B_\sigma(x)$ be the open ball in \mathbb{R}^3 of radius σ centred at x . If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and for $q \in [2, 2_s^*)$, we have*

$$(3.7) \quad \sup_{x \in \mathbb{R}^3} \int_{B_\sigma(x)} |u_n|^q \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2, 2_s^*)$.

Proof of Theorem 1. Choose $\phi \in C_0^\infty(\mathbb{R}^3)$ and $\xi > 0$. Define a path $h : [0, 1] \rightarrow E$ by $h(\eta) = \eta \xi \phi$. When ξ is large enough, by lemma 2.4, we have $I_\lambda(h(1)) < 0$, $\|h(1)\|_E > \rho$ for small ρ and $\sup_{\eta \in [0,1]} I_\lambda(\gamma(\eta)) \leq c$ for some c independent of $\eta \in [0, 1]$. Define

$$c_\lambda = \inf_{\gamma \in \Gamma} \sup_{\eta \in [0,1]} I_\lambda(\gamma(\eta)),$$

where $\Gamma = \{\gamma | \gamma \in C([0, 1], E), \gamma(0) = 0, \gamma(1) = \xi\phi\}$. By Lemma 2.3, the mountain pass theorem holds and c_λ is a critical value of I_λ . Therefore, we can choose $\lambda_n \rightarrow 0$, and a sequence of critical points $\{u_n\} \subset E$ satisfying $I'_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq c$. By Theorem 3.2, up to a subsequence $u_n \rightharpoonup u$, and u is a critical point of I in $H^s(\mathbb{R}^3)$. We need to show that $u \neq 0$. Note that $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for any $q \in [2, 2_s^*]$, then

$$\begin{aligned} 0 = I'_{\lambda_n}(u_n)u_n &= \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u_n|^2 + u_n^2)dx + \int_{\mathbb{R}^3} \phi_{u_n}^t(x)u_n^2dx - \int_{\mathbb{R}^3} f(x, u_n)dx \\ &\quad + \lambda_n \int_{\mathbb{R}^3} V(x)u_n^2dx \\ &\geq \|u_n\|_{H^s}^2 - \varepsilon \|u_n\|_2^2 - C_\varepsilon \|u_n\|_p^p \\ &\geq C \|u_n\|_p^2 - C_\varepsilon \|u_n\|_p^p. \end{aligned}$$

Then $\|u_n\|_p \geq (\frac{C}{C_\varepsilon})^{\frac{1}{p-2}}$. If $u = 0$, then given any $x \in \mathbb{R}^3$, $u_n \rightharpoonup 0$ in $H^s(B_\sigma(x))$, since the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is locally compact for $q \in [1, 2_s^*)$, then $u_n \rightarrow 0$ in $L^p(B_\sigma(x))$. By Lemma 3.3, we have $u_n \rightarrow 0$ in $L^p(\mathbb{R}^3)$, which is a contradiction with $\|u_n\|_p \geq (\frac{C}{C_\varepsilon})^{\frac{1}{p-2}}$. Therefore, $u \neq 0$. The proof is complete.

REFERENCES

- [1] X. Feng, Nontrivial solution for Schrödinger-Poisson equations involving a fractional nonlocal operator via perturbation methods, Z. Angew. Math. Phys. 67 (2016) Art. 74, 10 pp.
- [2] K. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent, J. Differential Equations. 261 (2016) 3061-3106.
- [3] V. Benci, D. Fortunto, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods Nonlinear Anal. 11 (1998) 283-293.
- [4] K. Teng, Multiple solutions for a class of fractional Schrödinger equations in R^N , Nonlinear Anal. Real World Appl. 21 (2015) 76-86.
- [5] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521-573.
- [6] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237 (2006) 655-674.
- [7] N. Laskin, Fractional quantum mechanics, Phys. Rev. E 62 (2000) 3135-3145.
- [8] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66 (2002) 56-108.
- [9] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1970.
- [10] X. Liu, J. Liu, Z. Wang, Quasilinear elliptic equations via perturbation method, Proc. Amer. Math. Soc. 141 (2013) 253-263.
- [11] P.D'Avenia, G. Siciliano, M. Squassina, On fractional Choquard equation, Math. Models Methods Appl. Sci. 25 (2015) 1447-1476.
- [12] A. Cotsoilis, N.K. Tavoularis, Sharp Sobolev type inequalities for higher fractional derivatives, C. R. Math. Acad. Sci. Paris 335 (2002) 801-804.

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